Self-Intersection Times for Random Walk, and Random Walk in Random Scenery

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Self-Intersection Times for Random Walk, and Random Walk in Random Scenery in dimensions $d \geq 5$.

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Abstract

Let $\{S_k, k \geq 0\}$ be a symmetric random walk on $\mathbb{Z}^d$, and $\{\eta(x), x \in \mathbb{Z}^d\}$ an independent random field of centered i.i.d. with tail decay $P(\eta(x) > t) \approx \exp(-t^\alpha)$. We consider a Random Walk in Random Scenery, that is $X_n = \eta(S_0) + \cdots + \eta(S_n)$. We present asymptotics for the probability, over both randomness, that $\{X_n > n^\beta\}$ for $1/2 < \beta < 1$ and $1 < \alpha$. To obtain such asymptotics, we establish large deviations estimates for the the self-intersection local times process $\sum l_n^2(x)$, where $l_n(x) = \mathbb{1}\{S_0 = x\} + \cdots + \mathbb{1}\{S_n = x\}$ is the local time of the walk.

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1 Introduction.

We study transport in divergence free random velocity fields. For simplicity, we discretize both space and time and consider the simplest model of shear flow velocity fields:

$$\forall x, y \in \mathbb{Z} \times \mathbb{Z}^d, \qquad V(x, y) = \eta(y) e_x,$$

where e_x is a unit vector in the first coordinate of $\mathbb{Z}^{d+1}$, and $\{\eta(y), y \in \mathbb{Z}^d\}$ are i.i.d. real random variables. Thus, space consists of the sites of the cubic lattice $\mathbb{Z}^{d+1}$ and the direction of the shear flow is e_x. We wish to model a polluant evolving by two mechanisms:

- a passive transport by the velocity field;
- collisions with the other fluid particles; this is modeled by random symmetric increments $\{(\alpha_n, \beta_n) \in \mathbb{Z} \times \mathbb{Z}^d, n \in \mathbb{N}\}$, independent of the velocity field.

Thus, if $R_n \in \mathbb{Z} \times \mathbb{Z}^d$ is the polluant's position at time n, then

$$R_{n+1} - R_n = V(R_n) + (\alpha_{n+1}, \beta_{n+1}), \quad \text{and} \quad R_0 = (0,0). \tag{1}$$

When solving by induction for R_n, (1) yields

$$R_n = \left(\sum_{k=1}^{n} \alpha_k + \sum_{k=0}^{n} \eta(\sum_{i=1}^{k} \beta_i), \sum_{k=1}^{n} \beta_k \right). \tag{2}$$

The sum $\beta_1 + \cdots + \beta_n$ is denoted by S_n, and called the Random Walk (RW). The displacement along e_x consists of two independent parts: a sum of i.i.d. random variables $\alpha_1 + \cdots + \alpha_n$, and a sum of *dependent* random variables $\eta(S_0) + \cdots + \eta(S_n)$, which we denote by X_n and call the Random Walk in Random Scenery (RWRS). Writing it in terms of local times of the RW $\{S_n, n \in \mathbb{N}\}$, we get

$$X_n = \sum_{k=0}^{n} \eta(S_k) = \sum_{x \in \mathbb{Z}^d} l_n(x)\eta(x), \quad \text{where} \quad l_n(x) = \sum_{k=0}^{n} \mathbb{1}\{S_k = x\}. \tag{3}$$

The process $\{X_n, n \in \mathbb{N}\}$ was studied at about the same time by Kesten & Spitzer [11], Borodin [4, 5], and Matheron & de Marsily [15]. The fact that in dimension 1, $E[X_n^2] \sim n^{3/2}$ made the model popular and led the way to examples of *superdiffusive* behaviour. However, the *typical* behaviour of X_n resembles that of a sum of n independent variables all the more when dimension is large.

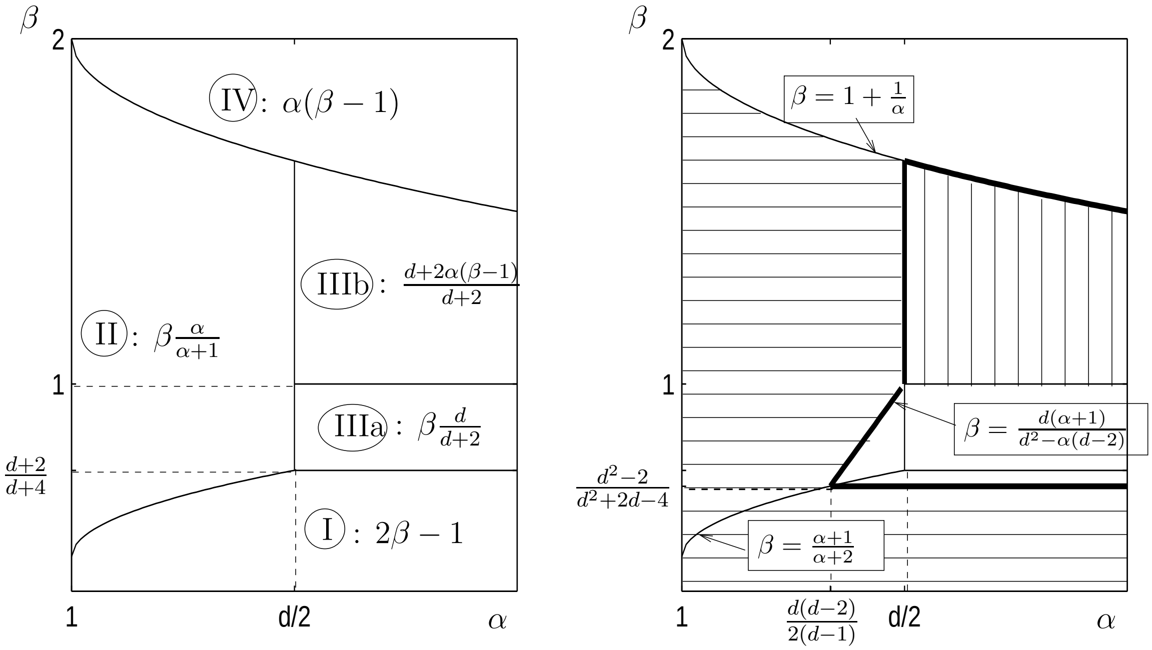
Our goal is to *estimate* the *probability* that X_n be *large*. By *probability*, we consider averages with respect to the two randomness, and $P = \mathbb{P}_0 \otimes P_\eta$, where $\mathbb{P}_0$ is the law of the nearest neighbor symmetric random walk $\{S_k, k \in \mathbb{N}\}$ on $\mathbb{Z}^d$ with $S_0 = 0$, and P_η is the law of the velocity field.

Now, when $d \geq 3$, Kesten and Spitzer established in [11] that $X_n/\sqrt{n}$ converges in law to a Gaussian variable. Thus, by *large*, we mean $\{X_n > n^\beta\}$ with $\beta > 1/2$. We expect $P(X_n > n^\beta) \approx \exp(-n^\zeta I)$ with constant rate $I > 0$, and we characterize in this work the exponent ζ. For this purpose, the only important feature of the η-variables is the α-exponent in the tail decay:

$$\lim_{t \to \infty} \frac{\log P_\eta(\eta(x) > t)}{t^\alpha} = -c, \quad \text{for a positive constant } c. \tag{4}$$

Let us now recall the classical estimates for $P(Y_1 + \cdots + Y_n > n^\beta)$, where $\beta > 1/2$ and the $\{Y_n, n \in \mathbb{N}\}$ are centered i.i.d. with tail decay $P(Y_n > t) \approx \exp(-t^a)$, with $a > 0$. There is a dichotomy between a "collective" and an "extreme" type of behaviour. In the former case, each variable contributes about the same, whereas in latter case, only one term exceeds the level n^β, when the others remain small. Thus, it is well known that $P(Y_1 + \cdots + Y_n > n^\beta) \sim \exp(-n^\zeta)$ with three regimes (since only the exponent ζ interests us, we have omitted all constants).

- When $\beta < 1$ and $\beta(2-a) < 1$, a small collective contribution yields $\zeta = 2\beta - 1$.
- When $\beta \geq 1$ and $a > 1$, a large collective contribution yields $\zeta = (\beta - 1)a + 1$.



Conjectured ζ-exponent. Proved Results.

Figure 1: ζ-exponent diagram

- When $\beta > 1/(2-a)$ and $a < 1$, an extreme contribution yields $\zeta = \beta a$.

For the RWRS, one expects a rich interplay between the scenery and the random walk. To get some intuition about the relationship between ζ and α, β, we propose simple scenarii leading to the left diagram in Figure 1. Here also, we focus on the exponent, and constants are omitted.

- **Region I**. No constraint is put on the walk. When $d \geq 3$, the range is of order n and sites of the range are typically visited once. Thus, $\{X_n > n^\beta\} \sim \{\eta_1 + \cdots + \eta_n > n^\beta\}$. When $\beta < 1$ the latter sum performs a moderate deviations of order n^β. Since the η-variable satisfy Cramer's condition, we obtain $P(X_n > n^\beta) \geq \exp(-n^{2\beta-1})$.

- **Region II, IV**. A few sites are visited often, so that $X_n \sim \eta(0) l_n(0)$. Now, using the tail behaviour of $\eta(x)$, and the fact that in $d \geq 3$, $l_n(x)$ is almost an exponential variable, we obtain

$$\begin{aligned} P\left[X_n \geq n^\beta\right] &\geq P\left[l_n(0)\eta(0) \geq n^\beta\right] \sim \sup_{k \leq n} \mathbb{P}_0\left[l_n(x) = k\right] P_\eta \left[\eta(x) \geq \frac{n^\beta}{k}\right] \\ &\sim \sup_{k \leq n} \exp(-k - (\frac{n^\beta}{k})^\alpha) \,. \end{aligned}$$

Now, the minimum of $k \mapsto k + n^{\beta\alpha}/k^\alpha$ is reached for $k^* = n^{\beta\alpha/(\alpha+1)}$. Since, we impose also that $k \leq n$, two different exponents prevail according to the value of β:

(II) $\beta < (\alpha+1)/\alpha$, and $\zeta = \beta\alpha/(\alpha+1)$. The RW spends a time of order $n^{\beta\alpha/(\alpha+1)}$ on favorite sites.

(IV) $\beta \geq (\alpha+1)/\alpha$, and $\zeta = \alpha(\beta-1)$. The RW spends a time of the order of n on one favorite site.

- **Region III.a, III.b**. The random walk is localized a time T in a ball B_r of radius r, with $r^2 \ll T$: this costs of the order of $\exp(-T/r^2)$. Then, during this period, each site of B_r is visited about T/r^d, and we further assume that $r^d \ll T$. Thus

$$P\left[X_n \geq n^\beta\right] \geq \exp(-\frac{T}{r^2}) P_\eta \left[\frac{1}{\sqrt{r^d}} \sum_{B_r} \eta_j \geq \frac{n^\beta r^{d/2}}{T}\right]. \tag{5}$$

Two different exponents prevail according to β:

(III.a) $\beta \leq 1$. The condition $1 \ll n^\beta r^{d/2}/T \ll r^{d/2}$ means that the sum of η-s performs a moderate (up to large) deviations and this costs of the order of $\exp(-n^{2\beta} r^d/T^2)$. When the two costs are equalized, we obtain that the walk is localized a time $T = n^\beta$ on a ball of radius $r = n^{\beta/(d+1)}$, and that $\zeta = d\beta/(d+2)$.

(III.b) $\beta > 1$. Here $T = n$ and we deal with a very large deviations for a sum of i.i.d. . This has a cost of order $\exp(-n^{\alpha(\beta-1)} r^d)$. Choosing r so that $n/r^2 = n^{\alpha(\beta-1)} r^d$, we obtain $\zeta = (d + 2\alpha(\beta-1))/(d+2)$. The condition $r \gg 1$ is equivalent to $\beta < 1 + \frac{1}{\alpha}$. The walk is localized all the time on a ball of radius r satisfying $r^{d+2} = n^{1-\alpha(\beta-1)}$.

The following regions have already been studied.

- $\alpha = +\infty$ (bounded scenery) and $\beta = 1$ in [1] (actually Brownian motion is considered there instead of RW);
- $\alpha = 2$ (Gaussian scenery) and $\beta \in [1, 1+1/\alpha]$ in [6, 7];
- Region III.b ($\alpha > d/2$, $1 \leq \beta < 1 + 1/\alpha$) in [10];
- $\beta = 1$ and $\alpha < d/2$ in [2].

In the first three cases, the precise decay rate is obtain (i.e. both the ζ-exponent and the rate I). In the case $\beta = 1$ and $\alpha < d/2$, distinct lower and upper bounds with the same exponent are given in [2]. Outside the diagram of Figure 1, in the region $0 < \alpha < 1$ and $\beta < \frac{1+\alpha}{2}$, in $d \geq 3$, precise estimates are established in [9].

This paper is devoted to regions I and II. Henceforth, we consider $d \geq 5$, unless explicitly mentioned.

Proposition 1.1 *Upper Bounds for the RWRS.*

1. *Region I. We assume (i)* $\beta \leq \min(\frac{\alpha+1}{\alpha+2}, \frac{d/2+1}{d/2+2})$ *and (ii)* $\beta < \frac{d^2-2}{d^2+2d-4}$. *There exists an explicit* y_0*, such that for* $y > y_0$*, there exists a constant* $\bar{c_1} > 0$*, and*

$$P(X_n \geq n^\beta y) \leq \exp(-\bar{c_1} n^{2\beta-1}). \tag{6}$$

2. *Region II. Let* $\alpha < d/2$, $\beta > \max\left\{\frac{\alpha+1}{\alpha+2}; \frac{d(\alpha+1)}{d^2-\alpha(d-2)}\right\}$. *For* $y > 0$, *there exists a constant* $\bar{c}_2 > 0$, *such that*

$$P(X_n \geq n^\beta y) \leq \exp(-\bar{c}_2 n^{\beta\alpha/(\alpha+1)}). \tag{7}$$

Remark 1.2 *In our context* $\frac{d^2-2}{d^2+2d-4}$ *is smaller than* $\frac{d/2+1}{d/2+2}$, *but we insisted on keeping the latter exponent in the definition of Region I, since the former is a technical artifact.*

We indicate below lower bounds for $P(X_n \geq n^\beta y)$, which prove that we have caught the correct rates of the logarithmic decay of $P(X_n \geq n^\beta y)$. These lower bounds are given under an additional symmetry assumption on the scenery, which is not crucial, but simplifies the proofs. Hence, we say that a real random variable is bell-shaped, if its law has a density with respect to Lebesgue which is even, and decreasing on $\mathbb{R}^+$.

Proposition 1.3 *Lower Bounds for the RWRS.*
Assume $d \geq 3$, *and that the random variables* $\{\eta(x), x \in \mathbb{Z}^d\}$ *are bell-shaped.*

1. *Region I. Let* $1 \geq \beta > 1/2$. *For all* $y > 0$, *there exists a constant* $\underline{c}_1 > 0$, *such that*

$$P(X_n \geq n^\beta y) \geq \exp(-\underline{c}_1 n^{2\beta-1}). \tag{8}$$

2. *Region II. Let* $\beta \leq 1 + 1/\alpha$. *For all* $y > 0$, *there exists a constant* $\underline{c}_2 > 0$, *such that*

$$P(X_n \geq n^\beta y) \geq \exp(-\underline{c}_2 n^{\beta\alpha/(\alpha+1)}). \tag{9}$$

The right diagram in Figure 1 summarizes the logarithmic decay rate of $P\left[X_n \geq n^\beta y\right]$. The hatched areas are explored territories (bold lines excluded), and the horizontal stripes correspond to Propositions 1.1 and 1.3.

In the process of establishing Proposition(1.1), one faces the problem of evaluating the chances the random walk visits often the same sites. More precisely, a crucial quantity is the *self-intersection local time process* (SILT):

$$\Sigma_n^2 = \sum_{x \in \mathbb{Z}^d} l_n^2(x) = n + 1 + 2 \sum_{0 \leq k < k' \leq n} 1\!\!1\{S_k = S_{k'}\}. \tag{10}$$

It is expected that Σ_n^2 would show up in the study of RWRS. Indeed, Σ_n^2 is the variance of X_n when averaged over P_η. If we assume for a moment that the η-variables are standard Gaussian, then conditionally on the random walk, X_n is a Gaussian variable with variance Σ_n^2, so that

$$P_\eta\left(\sum_{x \in \mathbb{Z}^d} \eta(x) l_n(x) > n^\beta\right) \leq \exp\left(-\frac{n^{2\beta}}{2\sum_{x \in \mathbb{Z}^d} l_n^2(x)}\right) \tag{11}$$

It is well known that (11) holds for any tail behaviour (4) with $\alpha \geq 2$. Now, if we average with respect to the random walk law, then for any $\gamma > 0$

$$P(X_n > n^\beta) \;\leq\; \mathbb{E}_0\left[\exp\left(-\frac{n^{2\beta}}{2\sum_{x \in \mathbb{Z}^d} l_n^2(x)}\right)\right]$$

$$\leq \exp(-n^{2\beta-\gamma}) + \mathbb{P}_0\left(\sum_{x\in\mathbb{Z}^d} l_n^2(x) > n^\gamma\right). \tag{12}$$

Hence, at least for large α, we have to evaluate the logarithmic decay of quantities such as $\mathbb{P}_0\left(\sum_{x\in\mathbb{Z}^d} l_n^2(x) > n^\gamma\right)$. Note first that for $d \geq 3$, and $n \to \infty$,

$$\mathbb{E}_0\left[\sum_{x\in\mathbb{Z}^d} l_n^2(x)\right] \simeq n(2G_d(0)-1), \tag{13}$$

where G_d is the Green kernel

$$G_d(x) \triangleq \mathbb{E}_0\left[l_\infty(x)\right].$$

Therefore, we have to take $\gamma \geq 1$ to be in a large deviations scaling. For large deviations of SILT in $d=1$, we refer the reader to Mansmann [14], and Chen & Li [8], while in $d=2$, this problem is treated in Bass & Chen [3].

We first present large deviations estimates for the SILT.

Proposition 1.4 *Assume $d \geq 5$. For $y > 1 + 2\sum_{x\in\mathbb{Z}^d} G_d(x)^2$, there are positive constants $\underline{c}, \bar{c}$ such that*

$$\exp(-\bar{c}\sqrt{n}) \geq \mathbb{P}_0\left[\sum_{x\in\mathbb{Z}^d} l_n^2(x) \geq ny\right] \geq \exp(-\underline{c}\sqrt{n}). \tag{14}$$

Proposition 1.4 is a corollary of the next result where we prove that the main contribution in the estimates comes from the region where the local time is of order $\sqrt{n}$.

Proposition 1.5 *1. For $\epsilon > 0$, and $y > 1 + 2\sum_{x\in\mathbb{Z}^d} G_d(x)^2$,*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_0\left[\sum_{x: l_n(x) \leq n^{1/2-\epsilon}} l_n^2(x) \geq ny\right] = -\infty. \tag{15}$$

2. For $y > 0$ and $\epsilon > 0$, there exists a constant $\tilde{c} > 0$, such that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_0\left[\sum_{x: l_n(x) > n^{1/2-\epsilon}} l_n^2(x) \geq ny\right] \leq -\tilde{c}. \tag{16}$$

Let us give some heuristics on the proof of Proposition 1.5. First of all, we decompose Σ_n^2 using the level sets of the local time. Note that it is not useful to consider $\{x : l_n(x) \gg \sqrt{n}\}$, since $l_n(x)$ is bounded by an exponential variable. Now, for a subdivision $\{b_i\}_{i\in\mathbb{N}}$ of $[0, 1/2]$, let $\mathcal{D}_{b_i} = \left\{x \in \mathbb{Z}^d, n^{b_i} \leq l_n(x) < n^{b_{i+1}}\right\}$. Denoting by $|\Lambda|$ the number of sites in $\Lambda \subset \mathbb{Z}^d$, we then have

$$\Sigma_n^2 = \sum_i \sum_{x\in\mathcal{D}_{b_i}} l_n^2(x) \leq \sum_i n^{2b_{i+1}} |\mathcal{D}_{b_i}|.$$

Hence, choosing $(y_{b_i})_{i\in\mathbb{N}}$ such that $\sum_i y_{b_i} \leq y$,

$$\mathbb{P}_0\left[\Sigma_n^2 \geq ny\right] \leq \sum_i \mathbb{P}_0\left[\sum_{x\in\mathcal{D}_{b_i}} l_n^2(x) \geq ny_{b_i}\right] \leq \sum_i \mathbb{P}_0\left[|\mathcal{D}_{b_i}| \geq n^{1-2b_{i+1}} y_{b_i}\right].$$

A first estimate of the right hand term is given by Lemma 1.2 of [2].

Lemma 1.6 *Assume $d \geq 3$. There is a constant $\kappa_d > 0$ such that for any $\Lambda \subset \mathbb{Z}^d$, and any $t > 0$*

$$P\left[l_\infty(\Lambda) > t\right] \leq \exp\left(-\kappa_d \frac{t}{|\Lambda|^{2/d}}\right), \quad \textit{where} \quad l_\infty(\Lambda) = \sum_{x\in\Lambda} l_\infty(x)\,.$$

Hence, if we drop the index i, and set $b = b_{i+1} \approx b_i$, for $L = n^{1-2b} y_b$, we have

$$\begin{aligned}\mathbb{P}_0\left[|\mathcal{D}_b| \geq L\right] &\leq \sum_{\Lambda\subset]-n;n[^d, |\Lambda|=L} \mathbb{P}_0\left(\mathcal{D}_b = \Lambda, l_n(\Lambda) \geq n^b L\right) \\ &\leq (2n)^{dL} \exp(-\kappa_d y_b n^\zeta) \quad \text{with} \quad \zeta = b + (1-\frac{2}{d})(1-2b). \end{aligned} \tag{17}$$

Since $\zeta > 1/2$ when $b < 1/2$ and $d > 4$, this estimate would suffice if the combinatorial factor $(2n)^{dL}$ were negligible. This case corresponds to "large" b. For "small" b, we need to get rid of the combinatorial term. We propose a reduction to intersection local times of two independent random walks. Assume for a moment that in place of $\sum_{x\in\mathcal{D}_b} l_n^2(x)$, we were to deal with $\sum_{x\in\mathcal{D}_b} l_n(x)\tilde{l}_n(x)$, where $(\tilde{l}_n(x))_{x\in\mathbb{Z}^d}$ is an independent copy of $(l_n(x))_{x\in\mathbb{Z}^d}$. Then, using Lemma 1.6, we obtain

$$\mathbb{P}_0 \otimes \tilde{\mathbb{P}}_0\left[\sum_{x\in\mathcal{D}_b} l_n(x)\tilde{l}_n(x) \geq n y_b\right] \leq \mathbb{P}_0 \otimes \tilde{\mathbb{P}}_0\left[\tilde{l}_n(\mathcal{D}_b) \geq n^{1-b} y_b\right] \leq \mathbb{E}_0\left[\exp\left(-\kappa_d \frac{n^{1-b} y_b}{|\mathcal{D}_b|^{2/d}}\right)\right].$$

Now, the simplest upper bound on the volume of $\mathcal{D}_b$ is $n^b|\mathcal{D}_b| \leq \sum_{x\in\mathcal{D}_b} l_n(x) \leq n$, so that

$$\mathbb{P}_0\left[\sum_{x\in\mathcal{D}_b} l_n(x)\tilde{l}_n(x) \geq n y_b\right] \leq \exp(-\kappa_d y_b n^{1-b-2/d(1-b)})\,.$$

In this crude way, we have obtain a weaker bound than (17), but without the combinatorial term. This can be improved as we improve on the upper bound for the volume of $\mathcal{D}_b$.

The paper is organized as follows. In Section 2, we prove the lower bound in Proposition (1.4), and present an upper bound rougher than (14). Indeed, the first step in establishing (14) is to reduce SILT into intersection local times of two independent walks. However, this reduction only yields

$$\mathbb{P}_0\left[\sum_{x\in\mathbb{Z}^d} l_n^2(x) \geq ny\right] \leq \exp\left(-\bar{c}\frac{\sqrt{n}}{\sqrt{\log(n)}}\right). \tag{18}$$

For pedagogical reasons, we first prove (18), whereas its refinements are proved in Section 4. We gather the technical Lemmas in Section 3. The results of Section 3 are applied to the problem of large deviations for SILT in Section 4, to obtain Proposition 1.5. In Section 5, we treat the problem of large and moderate deviations estimates for the RWRS, and prove Proposition 1.1. Finally, the corresponding lower bounds (Proposition 1.3) are shown in Section 6.

2 Rough estimates for SILT

Lower Bound. For $k \in \mathbb{N}$, let $T_0^{(k)}$ be the k-th return time at 0:

$$T_0^{(0)} \triangleq 0\,,\ T_0^{(k)} \triangleq \inf\left\{n > T_0^{(k-1)}, S_n = 0\right\}\,.$$

For $y > 0$,

$$\begin{aligned}
P\left[\sum_{x\in\mathbb{Z}^d} l_n^2(x) \ge ny\right] &\ge P\left[l_n(0) \ge \lfloor\sqrt{ny}\rfloor + 1\right] = P\left[T_0^{(\lfloor\sqrt{ny}\rfloor)} \le n\right]\\
&\ge P\left[\forall k \in \{1,\cdots\lfloor\sqrt{ny}\rfloor\}, T_0^{(k)} - T_0^{(k-1)} \le \frac{n}{\lfloor\sqrt{ny}\rfloor}\right]\\
&\ge P\left[T_0 \le \frac{n}{\sqrt{ny}}\right]^{\sqrt{ny}}\\
&= \left(P\left[T_0 < \infty\right] - P\left[\frac{\sqrt{n}}{\sqrt{y}} < T_0 < \infty\right]\right)^{\sqrt{ny}}
\end{aligned}$$

This proves the lower bound since $\lim_{n\to\infty} P\left[\frac{\sqrt{n}}{y} < T_0 < \infty\right] = 0$, and $P(T_0 < \infty) < 1$ for $d \ge 3$.

Upper bound (18).

We write the proof of the upper bound for $n = 2^N$ since this simplifies notations. The trivial extension to general n is omitted. First, note that

$$\sum_{x\in\mathbb{Z}^d} l_n^2(x) = n + 1 + 2Z^{(0)}, \quad \text{with} \quad Z^{(0)} = \sum_{0\le k<k'\le n} 1\!\!1\{S_k = S_{k'}\}\,.$$

The idea is to reduce $Z^{(0)}$ to the intersection times of two independent random walks stemming from $S_{n/2}$, as in Le Gall [13]. Then, we use moment estimates for intersection of independent random walks obtained in [12].

We divide $Z^{(0)}$ into

$$Z^{(0)} = Z_1^{(1)} + Z_2^{(1)} + \sum_{0\le k<2^{N-1}<k'\le 2^N} 1\!\!1\{S_k = S_{k'}\},$$

with,

$$Z_1^{(1)} = \sum_{k,k'} 1\!\!1\{0 \le k < k' \le 2^{N-1}\}\ 1\!\!1\{S_k = S_{k'}\},$$

and,

$$Z_2^{(1)} = \sum_{k,k'} 1\!\!1\{2^{N-1} \le k < k' \le 2^N\}\ 1\!\!1\{S_k = S_{k'}\}.$$

Thus, we can define two independent walks for times $k \in \{0, \ldots, 2^{N-1}\}$

$$S_{k,1} = S_{2^{N-1}} - S_{2^{N-1}-k}, \quad \text{and} \quad S_{k,2} = S_{2^{N-1}} - S_{2^{N-1}+k}.$$

Finally, we obtain $Z^{(0)} \le Z_1^{(1)} + Z_2^{(1)} + I_1^{(1)}$ with for $i = 1, 2$

$$Z_i^{(1)} = \sum_{k,k'} 1\!\!1\{0 \le k < k' \le 2^{N-1}\}\, 1\!\!1\{S_{k,i} = S_{k',i}\}, \quad \text{and} \quad I_1^{(1)} = \sum_{x \in \mathbb{Z}^d} l_{2^{N-1},1}(x) l_{2^{N-1},2}(x)\,,$$

where for $i = 1, 2$, $(l_{k,i}(x))_{k\in\mathbb{N}}$ denotes the local times of the random walk $(S_{k,i})_{k\in\mathbb{N}}$. Iterating this procedure, we get

$$Z^{(0)} \le \sum_{l=1}^{N-1} \sum_{k=1}^{2^{l-1}} I_k^{(l)}\,, \tag{19}$$

where for each $l \in \{1, \cdots, N-1\}$, the random variables $(I_k^{(l)}; 1 \le k \le 2^{l-1})$ are i.i.d. , and are distributed as $\sum_{x\in\mathbb{Z}^d} l_{2^{N-l}}(x)\tilde{l}_{2^{N-l}}(x)$, $(\tilde{l}_n(x), x \in \mathbb{Z}^d)$ being an independent copy of $(l_n(x), x \in \mathbb{Z}^d)$. Hence, to prove (18), it is enough to prove that for all $y > \sum_{x\in\mathbb{Z}^d} G_d(x)^2$,

$$\mathbb{P}_0\left[Z^{(0)} \ge 2^N y\right] \le \exp\left(-\bar{c}\frac{\sqrt{2^N y}}{N}\right)\,. \tag{20}$$

Now, for any $\{y_1, \ldots, y_{N-1}\}$ positive reals summing up to $\bar{y} \le y$, we have

$$\mathbb{P}_0(Z^{(0)} \ge 2^N y) \le \sum_{l=1}^{N-1} \mathbb{P}_0\left(\sum_{k=1}^{2^{l-1}} I_k^{(l)} \ge 2^N y_l\right)\,.$$

The strategy is now the following:

- When l is *small*, we bound each term of $\{I_k^{(l)}, k = 1, \ldots, 2^{l-1}\}$ by a sequence $\{I_k, k = 1, \ldots, 2^{l-1}\}$ of i.i.d. random variables distributed as I_∞, where

$$I_\infty = \sum_{x\in\mathbb{Z}^d} l_\infty(x)\tilde{l}_\infty(x)\,,$$

  $(\tilde{l}_\infty(x))_{x\in\mathbb{Z}^d}$ being an independent copy of $(l_\infty(x))_{x\in\mathbb{Z}^d}$. The estimate $\mathbb{P}_0(I_\infty > t) \le \exp(-\kappa_s\sqrt{t})$ is proved in [12]. Thus, in each generation, we are dealing with a sum of independent variables with the right streched exponential tails.

- When l is *large*, we use the trivial bound $I_k^{(l)} \le 2^{2(N-l)}$, and the classical Cramer's estimates.

First, we choose the $\{y_k\}$. Note that for $d \ge 5$, $m_1 = \mathbb{E}_0[I_\infty] = \sum_{x\in\mathbb{Z}^d} G_d(x)^2 < \infty$, so that we have to choose y_l such that

$$2^N y_l > 2^{l-1} m_1 \ge 2^{l-1}\mathbb{E}_0[I_k^{(l)}]\,.$$

A convenient choice is the following: for $l^* = 9N/10$,

- $y_l = y/(2N)$ for $l < l^*$.
- $y_l = y/2^{N-l+1}$, for $l \ge l^*$.

Thus, $\sum_{l=1}^{N-1} y_l \leq (l^*/(2N)+1/2)y < y$, and we obtain $\mathbb{P}_0(Z^{(0)} > 2^N y) \leq R_1 + R_2$ with

$$R_1 = \sum_{l<l^*} \mathbb{P}_0 \left(\sum_{k=1}^{2^{l-1}} I_k^{(l)} \geq \frac{2^N y}{2N} \right), \quad \text{and} \quad R_2 = \sum_{l \geq l^*} \mathbb{P}_0 \left(\sum_{k=1}^{2^{l-1}} \bar{I}_k^{(l)} \geq 2^{l-1}(y - m_1) \right),$$

where $\bar{I}_k^{(l)} = I_k^{(l)} - E[I_k^{(l)}]$.

Case $l < l^*$. We take advantage of the small size of the l-th generation to compare $I_k^{(l)}$ with I_∞ and use the bounds of [12]. Set $z_N = 2^N/N$ and $J_k = I_k^{(l)}\ \mathbb{I}\{I_k^{(l)} < z_N\}$, so that

$$\mathbb{P}_0 \left(\sum_{k=1}^{2^{l-1}} I_k^{(l)} \geq 2^N y_l \right) \leq 2^{l-1} \mathbb{P}_0 \left(I_k^{(l)} \geq z_N \right) + \mathbb{P}_0 \left(\sum_{k=1}^{2^{l-1}} J_k \geq 2^N y_l \right).$$

For any $\lambda > 0$,

$$\mathbb{P}_0 \left(\sum_{k=1}^{2^{l-1}} J_k \geq 2^N y_l \right) \leq e^{-\lambda 2^N y_l} \left(\mathbb{E}_0[e^{\lambda J_k}] \right)^{2^{l-1}}.$$

Now, using [12],

$$\mathbb{E}_0[e^{\lambda J_k}] = 1 + \int_0^{z_N} \lambda e^{\lambda u} P(J_k > u) du \leq 1 + \lambda \int_0^{z_N} e^{\lambda u - \kappa_s \sqrt{u}} du.$$

We choose $\lambda = \kappa_s / 2\sqrt{z_N}$ so that $\kappa_s \sqrt{u} \geq 2\lambda u$ for $u \leq z_N$. For such a choice, there is $c_0 > 0$ such that $\mathbb{E}_0[e^{\lambda J_k}] \leq 1 + c_0 \lambda$, and

$$\left(\mathbb{E}_0[e^{\lambda J_k}] \right)^{2^l} \leq e^{\lambda c_0 2^l}, \quad \text{for} \quad \lambda = \frac{\kappa_s}{2} \sqrt{\frac{N}{2^N}}.$$

Finally,

$$\mathbb{P}_0 \left(\sum_{k=1}^{2^{l-1}} I_k^{(l)} \geq \frac{2^N y}{2N} \right) \leq 2^l \exp(-\kappa_s \sqrt{\frac{2^N}{N}}) + \exp(-\frac{\kappa_s y}{4N} \sqrt{N 2^N} + \frac{c_0 \kappa_s 2^l}{2} \sqrt{\frac{N}{2^N}}).$$

This provides the desired bound as long as $2^l \ll 2^N/N$, which is always the case for $l < l^*$.

Case $l \geq l^*$. Note that for all l, $I^{(l)} \leq 2^{2(N-l)}$. Hence, for large l, we use large deviations estimates for sums of i.i.d. "small" random variables. More precisely, using Markov inequality, for all $\lambda > 0$,

$$\mathbb{P}_0 \left[\sum_{j=1}^{2^{l-1}} \bar{I}_k^{(l)} \geq 2^{l-1}(y - m_1) \right] \leq \exp\left(-\lambda 2^{l-1}(y - m_1)\right) \mathbb{E}_0 \left[\exp\left(\lambda \bar{I}^{(l)}\right) \right]^{2^{l-1}}.$$

We choose $\lambda \leq 1/2^{2(N-l)}$ and use the fact that $\exp(x) \leq 1 + x + 2x^2$ for $|x| \leq 1$, to obtain

$$\mathbb{E}_0 \left[\exp\left(\lambda \bar{I}_k^{(l)} \right) \right] \leq 1 + 2\lambda^2 E[(\bar{I}_k^{(l)})^2] \leq 1 + 2 m_2 \lambda^2,$$

where $m_2 = \mathbb{E}_0 \left[I_\infty^2\right] < \infty$ by [12]. Thus,

$$R_2 \leq \sum_{l \geq l^*} \exp\left(-2^{l-1}\lambda\left((y-m_1) - 2m_2\lambda\right)\right) . \tag{21}$$

Thus, we need $(y - m_1) > 2m_2\lambda$ and $\lambda \leq 1/2^{2(N-l)}$ for $l \geq l^*$. For $l^* = 9N/10$, it is enough to choose $\lambda = 2^{-N/5}$. ■

3 Technical Lemmas.

This Section provides the key estimates for the probability that $\sum l_n^p(x)$ be large. In a first reading of Lemmas 3.2 and 3.3, we suggest the reader to think of the case $p = 2$, $\gamma = 1$, $\zeta = 1/2$, on which relies our results on SILT. The cases $p \neq 2$ are needed for the proofs of the moderate deviations for the RWRS, but involve no additional ideas.

We begin with a simple improvement of Lemma 1.6.

Lemma 3.1 *Assume $d \geq 3$. There exists a constant $\kappa_d > 0$ such that for any $t > 0$, $L \geq 1$,*

$$\mathbb{P}_0\left[\left|\{x : l_n(x) \geq t\}\right| \geq L\right] \leq (2n)^{dL} \exp(-\kappa_d t L^{1-2/d}) . \tag{22}$$

Proof: The proof is a simple application of Lemma 1.6 of [2].

$$\begin{aligned} P(\left|\{x : l_n(x) \geq t\}\right| \geq L) &\leq \sum_{\Lambda \subset]-n;n[^d; |\Lambda| = L} P(\forall x \in \Lambda, l_n(x) \geq t) \\ &\leq \sum_{\Lambda \subset]-n;n[^d; |\Lambda| = L} P(l_n(\Lambda) \geq Lt) \leq n^{dL} \exp(-\kappa_d t L^{1-2/d}). \end{aligned} \tag{23}$$

■

As a corollary of Lemma 3.1, we obtain the following estimates for the regions where the local times are *large*. We recall that for $p > 1$, we denote by $p^* := p/(1-p)$ the conjugate exponent.

Lemma 3.2 *Assume $d \geq 3$, and fix positive real numbers $a, b, \gamma, \zeta, p, y$, with*

$$\frac{\zeta}{d/2} \leq b < a, \quad \textit{and define} \quad \mathcal{D} = \left\{x : n^b \leq l_n(x) \leq n^a\right\} .$$

We assume either of the following two conditions.

(i) $p \geq (d/2)^*$; $\zeta < \frac{\gamma}{p(2/d)+1}$; $a < \frac{\gamma - \zeta(d/2)^*}{p - (d/2)^*}$.

(ii) $1 < p < (d/2)^*$; $b > \frac{\zeta(d/2)^* - \gamma}{(d/2)^* - p}$.

Then, for a constant c (depending on $a, b, p, \gamma, \zeta, y$) and for n large enough,

$$\mathbb{P}_0\left[\sum_{x \in \mathcal{D}} l_n^p(x) \geq n^\gamma y\right] \leq \exp(-cn^\zeta) . \tag{24}$$

Proof: The strategy is to slice the above sum according to the level sets of the local times. Thus, we decompose $\mathcal{D}$ into a finite number M of regions. For $i = 0, \dots, M$, let

$$\mathcal{D}_i = \left\{x : n^{b_i} \le l_n(x) < n^{b_{i+1}}\right\}, \quad \text{where} \quad b = b_0 < b_1 < \cdots < b_M, \quad b_M \ge a. \tag{25}$$

M and the sequence $\{b_i; 0 \le i \le M\}$ will be chosen later. Then,

$$\mathbb{P}_0\left[\sum_{x\in\mathcal{D}} l_n^p(x) \ge n^\gamma y\right] \le \sum_{i=0}^{M-1} \mathbb{P}_0\left[\sum_{x\in\mathcal{D}_i} l_n^p(x) \ge \frac{n^\gamma y}{M}\right] \tag{26}$$

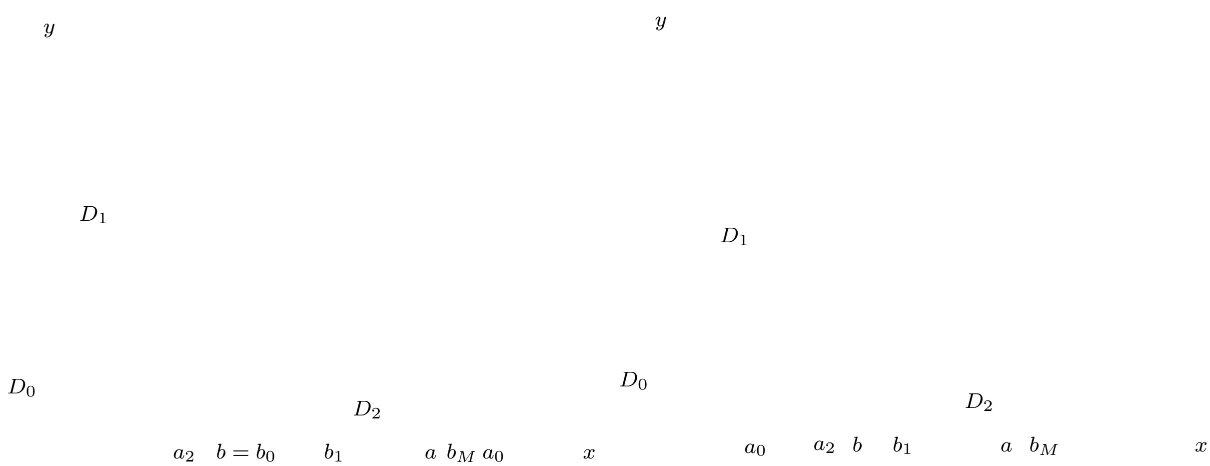
$$\le \sum_{i=0}^{M-1} \mathbb{P}_0\left[|\mathcal{D}_i| \ge \frac{n^{\gamma - pb_{i+1}} y}{M}\right]. \tag{27}$$

We now use Lemma 3.1 with $t = n^{b_i}$ and $L = n^{\gamma - pb_{i+1}} y/M$ to get

$$\mathbb{P}_0\left[\sum_{x\in\mathcal{D}} l_n^p(x) \ge n^\gamma y\right] \le \sum_{i=0}^{M-1} n^{dn^{\gamma - pb_{i+1}} y/M} \exp\left(-\kappa_d n^{b_i + (1-2/d)(\gamma - pb_{i+1})} (y/M)^{1-2/d}\right). \tag{28}$$

To conclude, it is now enough to check that we can find a finite sequence $(b_i, 0 \le i \le M)$, such that $b_0 = b$, $b_M > a$ and satisfying the constraints

$$\begin{cases} \gamma - pb_{i+1} < b_i + (1-2/d)(\gamma - pb_{i+1}) \\ \zeta \le b_i + (1-2/d)(\gamma - pb_{i+1}) \\ b_i < b_{i+1} \end{cases} \Leftrightarrow \begin{cases} b_{i+1} > \frac{\gamma}{p} - \frac{d}{2p} b_i & (C_2) \\ b_{i+1} \le \frac{\gamma}{p} + \frac{d}{p(d-2)}(b_i - \zeta) & (C_1) \\ b_{i+1} > b_i & (C_0) \end{cases}. \tag{29}$$



Case $p \ge \frac{d}{d-2}$ Case $p < \frac{d}{d-2}$

Figure 2: Construction of $(b_i, 0 \le i \le M)$ for $\mathcal{D}$

Let D_0 be the line $y = x$, D_1 be the line $y = \frac{\gamma}{p} + \frac{d}{p(d-2)}(x - \zeta)$, and D_2 the line $y = \frac{\gamma}{p} - \frac{d}{2p}x$.
Case $p \ge (d/2)^*$: In that case, the slope of D_1 is less than 1. Let a_0 (resp. a_2) be the abscisse of the intersection of D_1 with D_0 (resp. D_2)

$$a_0 = \frac{\gamma - \zeta(d/2)^*}{p - (d/2)^*}, \quad a_2 = \frac{\zeta}{(d/2)}.$$

Then, the region of constraints is non empty (see figure 2) if and only if

$$a_2 < a_0 \Leftrightarrow \zeta < \frac{\gamma}{1+2p/d}.$$

In that case, it is always possible to construct a finite sequence $(b_i)_{0\le i\le M}$ satisfying the constraints $(C_0),(C_1),(C_2)$ and $b_0 = b$, $b_M \ge a$, as soon as $b \ge a_2$ and $a < a_0$. A possible choice is to take $b_{i+1} = \frac{\gamma}{p} + \frac{d}{p(d-2)}(b_i - \zeta)$, M being defined by $b_{M-1} < a \le b_M$.
<u>Case $p < (d/2)^*$</u>: In that case, the slope of D_1 is strictly greater than 1, and the region of constraints is never empty. It is always possible to construct a finite sequence $(b_i)_{0\le i\le M}$ satisfying the constraints $(C_0),(C_1),(C_2)$ as soon as $b > a_0$, and $b \ge a_2$. A possible choice is to take $b_{i+1} = \frac{\gamma}{p} + \frac{d}{p(d-2)}(b_i - \zeta)$, M being defined by $b_{M-1} < a \le b_M$. ■

We deal now with the regions where the local times are *small*.

Lemma 3.3 *Assume $d \ge 5$, and fix positive real numbers b, ζ, y. Let*

$$\underline{\mathcal{D}_b} = \left\{ x \in \mathbb{Z}^d : l_n(x) \le n^b \right\}.$$

Assume that

$$\gamma \ge 1; \quad \zeta < \gamma - b - \frac{2}{d}(1-b); \quad \textit{and} \qquad \begin{cases} y > 0 & \textit{if } \gamma > 1, \\ y > 1 + 2\sum_{x\in\mathbb{Z}^d} G_d(x)^2 & \textit{if } \gamma = 1. \end{cases} \tag{30}$$

Then, for a constant c (depending on b, γ, ζ, y) and for n large enough, we have

$$\mathbb{P}_0\left[\sum_{x\in\underline{\mathcal{D}_b}} l_n^2(x) \ge n^\gamma y \right] \le \exp(-cn^\zeta). \tag{31}$$

Proof: We again perform the decomposition in terms of level sets with $\mathcal{D}_i$ as in (25). However, we are now in a region where the estimate (22) is useless since the combinatorial factor is dominant. To overcome this problem, we rewrite SILT in Proposition 1.4 in terms of intersections of independent random walks, as explained in the introduction. We assume from now on, that n is a power of 2, $n = 2^N$. As in Proposition 1.4,

$$\sum_{x\in\underline{\mathcal{D}_b}} l_n^2(x) \le n + 1 + 2Z^{(0)}, \quad \text{with} \quad Z^{(0)} = \sum_{x\in\underline{\mathcal{D}_b}} \sum_{0\le k<k'\le 2^N} 1\!\!1\{S_k = S_{k'} = x\}. \tag{32}$$

Now,

$$\begin{aligned} Z^{(0)} \;\le\; & \sum_x 1\!\!1\{l_{2^{N-1}}(x) \le 2^{Nb}\} \sum_{0\le k<k'\le 2^{N-1}} 1\!\!1\{S_k = S_{k'} = x\} \\ & + \sum_x 1\!\!1\{l_{2^N}(x) - l_{2^{N-1}}(x) \le 2^{Nb}\} \sum_{2^{N-1}\le k<k'\le 2^N} 1\!\!1\{S_k = S_{k'} = x\} \\ & + \sum_x 1\!\!1\{l_{2^{N-1}}(x) \le 2^{Nb}\} \sum_{0\le k\le 2^{N-1}\le k'\le 2^N} 1\!\!1\{S_k = S_{k'} = x\} \\ \triangleq\; & Z_1^{(1)} + Z_2^{(1)} + J_1^{(1)}. \end{aligned}$$

With the same notations than in the proof of Proposition 1.4, for $i=1,2$

$$Z_i^{(1)} = \sum_y \mathbb{1}\{S_{2^{N-1}} = y\} \sum_x \mathbb{1}\{l_{2^{N-1},i}(y-x) \le 2^{Nb}\} \sum_{0\le k<k'\le 2^{N-1}} \mathbb{1}\{S_{k,i} = S_{k',i} = y-x\}\,.$$

Changing x in $y-x$ in the second summation, we obtain for $i=1,2$

$$Z_i^{(1)} = \sum_x \mathbb{1}\{l_{2^{N-1},i}(x) \le 2^{Nb}\} \sum_{0\le k<k'\le 2^{N-1}} \mathbb{1}\{S_{k,i} = S_{k',i} = x\}\,.$$

The self-intersection times of the two independent strands is denoted

$$J_1^{(1)} = \sum_x \mathbb{1}\{l_{2^{N-1},1}(x) \le 2^{Nb}\} l_{2^{N-1},1}(x) l_{2^{N-1},2}(x)\,.$$

Iterating this procedure, we get

$$Z^{(0)} \le \sum_{l=1}^{N-1} \sum_{k=1}^{2^{l-1}} J_k^{(l)}\,, \tag{33}$$

where for each $l \in \{1,\cdots,N-1\}$, the random variables $\{J_k^{(l)}; 1\le k\le 2^{l-1}\}$ are i.i.d. , and are distributed as a variable, say $J^{(l)}$, with

$$J^{(l)} = \sum_{x: l_{2^{N-l}}(x) \le 2^{Nb}} l_{2^{N-l}}(x) \tilde{l}_{2^{N-l}}(x),$$

where $\{\tilde{l}_n(x), x\in\mathbb{Z}^d\}$ is an independent copy of $\{l_n(x), x\in\mathbb{Z}^d\}$. Now, note that

$$\mathbb{E}_0\left[\sum_{l=1}^{N} \sum_{k=1}^{2^{l-1}} J_k^{(l)}\right] \le 2^N \sum_x G_d(x)^2.$$

Hence, if $\bar{J_k}^{(l)} = J_k^{(l)} - \mathbb{E}_0[J_k^{(l)}]$,

$$\mathbb{P}_0\left[\sum_{x\in\underline{\mathcal{D}_b}} l_n^2(x) \ge n^\gamma y\right] \le \mathbb{P}_0\left[\sum_{l=1}^{N} \sum_{k=1}^{2^{l-1}} \bar{J_k}^{(l)} \ge \frac{n^\gamma y - n - 1}{2} - n \sum_{x\in\mathbb{Z}^d} G_d^2(x)\right].$$

Thus, we need to prove that there exists a constant c such that $\mathbb{P}_0\left[\sum_{l=1}^N \sum_{k=1}^{2^{l-1}} \bar{J}_k^{(l)} \ge 2^{N\gamma} y\right] \le \exp(-c2^{N\zeta})$. Now

$$\mathbb{P}_0\left[\sum_{l=1}^{N} \sum_{k=1}^{2^{l-1}} \bar{J}_k^{(l)} \ge 2^{N\gamma} y\right] \le \sum_{l=1}^{N} \mathbb{P}_0\left[\sum_{k=1}^{2^{l-1}} \bar{J}_k^{(l)} > \frac{2^{N\gamma} y}{N}\right]. \tag{34}$$

We wish to use Cramer's estimates, so that we need the existence of some exponential moments for the $J_k^{(l)}$. For this purpose, we choose $\{b_i, i=0,\ldots,M\}$ a regular subdivision of $[0, b]$ of mesh $\delta > 0$ to be chosen later, such that $b = 0$, $b_M = M\delta \geq b > (M-1)\delta$. Let $\mathcal{D}_i = \{x; 2^{Nb_i} \leq l_{2^{N-l}}(x) < 2^{Nb_{i+1}}\}$. For each l and k, and any $u > 0$, we have, using Lemma 1.6, and independence between l and $\tilde{l}$,

$$
\begin{aligned}
\mathbb{P}_0(J^{(l)} > u) &\leq \sum_{i=0}^{M-1} \mathbb{P}_0\left(\sum_{\mathcal{D}_i} \tilde{l}_{2^{N-l}}(x) > \frac{u}{2^{Nb_{i+1}} M}\right) \\
&\leq \sum_{i=0}^{M-1} \mathbb{E}_0\left[\exp\left(-\kappa_d \frac{u}{2^{Nb_{i+1}} |\mathcal{D}_i|^{2/d} M}\right)\right]. \qquad (35)
\end{aligned}
$$

We now use a rough upper bound for $\mathcal{D}_i$: $2^{Nb_i}|\mathcal{D}_i| \leq 2^N$, to obtain

$$
\mathbb{P}_0(J^{(l)} > u) \leq \sum_{i=0}^{M-1} \exp\left(-\frac{\kappa_d u}{2^{N\zeta_i} M}\right) \leq M \exp\left(-\frac{\kappa_d u}{2^{N \max(\zeta_i)} M}\right), \qquad (36)
$$

with $\zeta_i = b_{i+1} + \frac{2}{d}(1 - b_i) = i\delta(1 - 2/d) + 2/d + \delta$. Thus,

$$
\max(\zeta_i) = M\delta(1 - 2/d) + 2/d + \delta \leq (b + \delta)(1 - 2/d) + 2/d + \delta,
$$

and for any $\epsilon > 0$, we can choose δ such that $\max(\zeta_i) \leq b + \frac{2}{d}(1 - b) + \frac{\epsilon}{2}$. Thus we have a constant C_ϵ such that

$$
\mathbb{P}_0(J^{(l)} > u) \leq C_\epsilon \exp(-\xi_N u), \quad \text{with} \quad \xi_N = \frac{\kappa_d}{M} 2^{-N(b+(1-b)2/d+\epsilon/2)}. \qquad (37)
$$

Note that this estimate is better than the estimate of [12]

$$
\mathbb{P}_0(J^{(l)} > u) \leq \mathbb{P}_0(I_\infty > u) \leq \exp(-\kappa_s \sqrt{u}), \qquad (38)
$$

only for $u > \kappa_s/\xi_N^2$. However, it permits us to consider exponential moment $E[\exp(\lambda J_k)]$ for $\lambda < \xi_N$. We now go back to the standard Cramer's method. For simplicity of notations, we drop the indices l and k when unambiguous. Returning now to (34), for any $0 \leq \lambda < \xi_N$,

$$
\mathbb{P}_0\left[\sum_{k=1}^{2^{l-1}} \bar{J}_k^{(l)} \geq \frac{2^{N\gamma} y}{N}\right] \leq \exp\left(-\lambda \frac{2^{N\gamma} y}{N}\right) \mathbb{E}_0\left[\exp(\lambda \bar{J})\right]^{2^{l-1}}. \qquad (39)
$$

Now, using the fact that $e^x \leq 1 + x + 2x^2$ for $x \leq 1$,

$$
\begin{aligned}
\mathbb{E}_0[e^{\lambda \bar{J}}] &= \mathbb{E}_0[e^{\lambda \bar{J}}\ 1\!\!1\{J < 1/\lambda\}] + \mathbb{E}_0[e^{\lambda \bar{J}}\ 1\!\!1\{J \geq 1/\lambda\}] \qquad (40) \\
&\leq \mathbb{E}_0[e^{\lambda \bar{J}}\ 1\!\!1\{J < 1/\lambda\}] + \mathbb{E}_0[e^{\lambda J}\ 1\!\!1\{J \geq 1/\lambda\}] \qquad (41) \\
&\leq \mathbb{E}_0\left[\left(1 + \lambda \bar{J} + 2\lambda^2 (\bar{J})^2\right)\ 1\!\!1\{J < 1/\lambda\}\right] + \mathbb{E}_0\left[e^{\lambda J}\ 1\!\!1\{J \geq 1/\lambda\}\right] \qquad (42) \\
&\leq 1 + \lambda \mathbb{E}_0\left[|\bar{J}|\ 1\!\!1\{J \geq 1/\lambda\}\right] + 2\lambda^2 \mathbb{E}_0\left[\bar{J}^2\right] + \mathbb{E}_0\left[e^{\lambda J}\ 1\!\!1\{J \geq 1/\lambda\}\right], \qquad (43)
\end{aligned}
$$

where we have used the fact that $\mathbb{E}_0\left[\bar{J}\right] = 0$. Now,

$$
\mathbb{E}_0\left[|\bar{J}|\ 1\!\!1\{J \geq 1/\lambda\}\right] \leq \mathbb{E}_0\left[(\bar{J})^2\right]^{1/2} \mathbb{P}_0\left[J \geq 1/\lambda\right]^{1/2} \leq \lambda \mathbb{E}_0\left[J^2\right] \leq \lambda \mathbb{E}_0(I_\infty^2).
$$

Note that by the results of [12], $\mathbb{E}_0(I_\infty^2) < \infty$. Hence, for some constant c,

$$\mathbb{E}_0[e^{\lambda \bar{J}}] \leq 1 + c\lambda^2 + \mathbb{E}_0\left[e^{\lambda J}\ \mathbb{I}\{J \geq 1/\lambda\}\right].$$

We now show that for some constant C, $E\left[e^{\lambda J}\ \mathbb{I}\{J \geq 1/\lambda\}\right] \leq C\lambda^2$. We decompose this last expectation into

$$\mathbb{E}_0\left[e^{\lambda J}\ \mathbb{I}\{J \geq 1/\lambda\}\right] = e^1\mathbb{P}_0(\lambda J \geq 1) + I \leq e^1\mathbb{E}_0[I_\infty^2]\lambda^2 + I,$$

with

$$I = \int_{1/\lambda}^{\infty} \lambda e^{\lambda u}\mathbb{P}_0(J \geq u)du, \quad \text{and we choose} \quad \lambda = \frac{\xi_N}{2\log(1/\xi_N^3)}.$$

To bound I, we use estimate (37), $\lambda < \xi_N/2$ and N large enough

$$\begin{aligned} I &\leq \int_{1/\lambda}^{\infty} \lambda e^{\lambda u - \xi_N u}du \leq \frac{2\lambda}{\xi_N}\int_{1/\lambda}^{\infty}(\xi_N/2)e^{-(\xi_N/2)u}du \\ &\leq \frac{2\lambda}{\xi_N}\exp(-\frac{\xi_N}{2\lambda}) \leq \frac{\xi_N^3}{\log(1/\xi_N^3)} \leq 4\xi_N\log(1/\xi_N^3)\lambda^2 \leq \lambda^2. \end{aligned} \tag{44}$$

Thus, there is a constant C such that

$$\mathbb{E}_0[\exp(\lambda\bar{J})] \leq 1 + C\lambda^2 \leq \exp(C\lambda^2),$$

which together with (39), yield

$$\mathbb{P}_0\left[\sum_{l=1}^{N-1}\sum_{k=1}^{2^{l-1}}\bar{J}_k \geq \frac{2^{N\gamma}y}{N}\right] \leq N\exp\left(-\frac{2^{N\gamma}y}{2N}\frac{\xi_N}{2\log(1/\xi_N^3)} + \frac{C\xi_N^2 2^l}{4\log^2(1/\xi_n^3)}\right) \tag{45}$$

$$\leq N\exp\left(-\frac{2^{N\gamma}y\xi_N}{8N\log(1/\xi_N^3)}\right), \tag{46}$$

where we used that $2^{N\gamma}y > 2CN\xi_N 2^l/\log(1/\xi_N^3)$ for any $l \leq N$ and N large enough, as soon as ϵ is chosen so that $\gamma - b - \frac{2}{d}(1-b) - \epsilon/2 > 0$. Now, we can use an extra $\epsilon/2$ to swallow the denominator $N\log(1/\xi_N^3)$ in the exponential, and the N factor in front of the exponential in (45). We obtain then for large enough N,

$$\mathbb{P}_0\left[\sum_{l=1}^{N-1}\sum_{k=1}^{2^{l-1}}\bar{J}_k^{(l)} \geq 2^{N\gamma}y\right] \leq \exp\left(-C2^{N\zeta}\right), \text{ with } \zeta = \gamma - b - \frac{2}{d}(1-b) - \epsilon. \tag{47}$$

■

4 Refined upper bound estimates for SILT.

In this Section, we prove Proposition 1.5. In the first Subsection, we apply Lemmas 3.2 and 3.3 to deal with the case $d \geq 6$, then we improve Lemma 3.3 to treat separetely the case $d = 5$, which we have added as a Lemma.

4.1 Proof of 1. of Proposition 1.5

The region $\{x : l_n(x) \le n^{1/2-\epsilon}\}$ is split into two regions

$$\mathcal{D} \triangleq \{x : n^b < l_n(x) \le n^{1/2-\epsilon}\}, \quad \text{and} \quad \underline{\mathcal{D}_b} \triangleq \{x : l_n(x) \le n^b\},$$

(b to be chosen later) so that for any $y > 1 + 2\sum_x G_d^2(x)$,

$$P\left[\sum_{x: l_n(x) \le n^{1/2-\epsilon}} l_n^2(x) \ge ny\right] \le P\left[\sum_{x\in\mathcal{D}} l_n^2(x) \ge ny_1\right] + P\left[\sum_{x\in\underline{\mathcal{D}_b}} l_n^2(x) \ge ny_2\right],$$

where $y_1 > 0$, $y_2 > 1 + 2\sum_x G_d^2(x)$, $y_1 + y_2 \le y$.

For the first region, we apply Lemma 3.2 with $p = 2$, $\gamma = 1$. Since $d \ge 4$, we are in Case (i) of Lemma 3.2, so that ζ has to verify $\zeta < \frac{d}{4+d}$. Note that for $d \ge 5$, $\frac{d}{4+d} > 1/2$, so that we can choose $\zeta = 1/2 + \eta$ where $\eta > 0$ is such that $1/2 + \eta < \frac{d}{4+d}$. We want now to take $a = 1/2 - \epsilon$, so that η has also to satisfy

$$a < \frac{\gamma(d-2) - d\zeta}{(d-2)p - d} \Leftrightarrow 1/2 - \epsilon < \frac{d - 2 - d(1/2+\eta)}{d-4} = \frac{1}{2} - \frac{d\eta}{d-4} \Leftrightarrow \eta < \frac{d-4}{d}\epsilon.$$

Thus for $b = \frac{2\zeta}{d} = \frac{1}{d} + \frac{2\eta}{d}$, Lemma 3.2 allows to conclude that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_0 \left[\sum_{x\in\mathcal{D}} l_n^2(x) \ge ny_1\right] = -\infty. \tag{48}$$

For the region $\underline{\mathcal{D}_b}$, we use Lemma 3.3, with $\gamma = 1$, $\zeta = \frac{1}{2} + \eta$, $b = \frac{1}{d} + \frac{2\eta}{d}$, $y = y_2$. We just have to check that we can find $\eta > 0$ such that

$$\zeta < \gamma - b - \frac{2}{d}(1-b) \Leftrightarrow (d^2 + 2d - 4)\eta < \frac{d^2}{2} - 3d + 2.$$

This is possible when $\frac{d^2}{2} - 3d + 2 > 0$, i.e. when $d \ge 6$.

For the case $d = 5$, we need a special treatment.

Lemma 4.1 *Assume $d = 5$. There exists $\epsilon > 0$ such that for $b \le 1/d + \epsilon$, for $y > 1 + 2\sum_x G_d^2(x)$,*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_0 \left(\sum_{\underline{\mathcal{D}_b}} l_n^2(x) > ny\right) = -\infty. \tag{49}$$

Proof: We use the same decomposition as in the proof of Lemma 3.3, up to inequality (35), where we use the rough estimate for $|\mathcal{D}_i|$ only for the *young* generations.
<u>Case $l \ge (2/d^2)N$</u>. We denote $\mathcal{D}_{i,k}^{(l)} = \{x : 2^{Nb_i} \le l_{2^{N-l},k}(x) < 2^{Nb_{i+1}}\}$, where we have associated $l_{2^{N-l},k}$ with the k-th variable $J_k^{(l)}$ appearing in (34). We actually add an index k

and l to make precise this correspondence. Hence, the rough bound is $|\mathcal{D}_{i,k}^{(l)}|2^{Nb_i} \leq 2^{N-l}$, so that after the appropriate changes, (37) reads: $\forall \delta > 0$, $\exists C$ such that $\forall l \geq (2/d^2)N$,

$$\mathbb{P}_0(J^{(l)} > u) \leq \exp(-C2^{-N\zeta}u), \quad \text{with} \quad \zeta = b(1-\frac{2}{d}) + \frac{2}{d}(1-\frac{2}{d^2}) + \delta. \tag{50}$$

By proceeding as in the proof of Lemma 3.3, we obtain that for $l \geq (2/d^2)N$,

$$\mathbb{P}_0\left[\sum_{k=1}^{2^{l-1}} \bar{J}_k^{(l)} \geq 2^N y\right] \leq \exp\left(-C2^{N(1-\zeta)}\right), \tag{51}$$

It is easy to check that one can find $\delta > 0$ such that $1-\zeta > 1/2$ as soon as $b < \frac{d-4}{2(d-2)} + \frac{4}{d^2(d-2)}$. Note that for $d = 5$, this last quantity is strictly bigger than $\frac{1}{d}$.
Case $l < (2/d^2)N$. The strategy for the *old* generations is to control the size of $\mathcal{D}_i$ by a boostrap-type argument. That is, if $\mathcal{D}_i$ is large, then $\sum_{\mathcal{D}_i} l_n^2(x)$ is large and Lemma 3.3 can be applied to control this term. Thus, for any γ_i,

$$\{|\mathcal{D}_{i,k}^{(l)}| > 2^{N(1-\gamma_i)}\} \subset \{\sum_{\mathcal{D}_{i,k}^{(l)}} l^2_{2^{N-l},k}(x) > 2^{N(1+2b_i-\gamma_i)}\}, \tag{52}$$

and we can invoke Lemma 3.3 to obtain a *good* γ_i. Before doing so, we go back to the right hand side of (34), and for a fixed l, we define

$$\mathcal{A} = \{\forall k = 1, \ldots, 2^{l-1}; \forall i = 1, \ldots, M; |\mathcal{D}_{i,k}^{(l)}| < 2^{N(1-\gamma_i)}\}$$

and perform the following partitioning

$$\mathbb{P}_0\left[\sum_{k=1}^{2^{l-1}} \bar{J}_k^{(l)} > \frac{2^{N\gamma}y}{N}\right] \leq \mathbb{P}_0\left[\mathcal{A}^c\right] + \mathbb{P}_0\left[\sum_{k=1}^{2^{l-1}} \bar{J}_k^{(l)} > \frac{2^{N\gamma}y}{N}, \mathcal{A}\right]. \tag{53}$$

Now, for $l < \frac{2}{d^2}N$, $N - l > N(1-\frac{2}{d^2})$, and $\mathcal{D}_{i,k}^{(l)} \subset \left\{x; l_{2^{N-l}}(x) \leq 2^{(N-l)b_{i+1}/(1-2/d^2)}\right\}$. Hence,

$$\begin{aligned} \mathbb{P}_0\left[\mathcal{A}^c\right] &\leq \sum_{k=1}^{2^{l-1}} \sum_i \mathbb{P}_0\left[|\mathcal{D}_{i,k}^{(l)}| \geq 2^{N(1-\gamma_i)}\right] & (54)\\ &\leq 2^{l-1}\sum_i \mathbb{P}_0\left[\sum_x \mathbb{1}_{\{l_{2^{N-l}}(x) \leq 2^{(N-l)\frac{b_{i+1}}{1-2/d^2}}\}} l^2_{2^{N-l}}(x) \geq 2^{(N-l)(1+2b_i-\gamma_i)}\right] & (55) \end{aligned}$$

We can now apply Lemma 3.3 at time 2^{N-l}, to bound $\mathbb{P}_0\left[\mathcal{A}^c\right]$ by $\exp(-2^{(N-l)\zeta})$. To obtain $\mathbb{P}_0\left[\mathcal{A}^c\right] \ll \exp(-C2^{N/2})$, we have to take $\zeta > (1/2)/(1-2/d^2)$. We have thus to choose γ_i in order to satisfy the following conditions

$$1 + 2b_i - \gamma_i > 1\,, \; \frac{1/2}{1-2/d^2} < 1 + 2b_i - \gamma_i - (1-\frac{2}{d})\frac{b_{i+1}}{1-2/d^2} - \frac{2}{d}. \tag{56}$$

We choose $\gamma_i = b_i(1+2/d)$. In that case, the two conditions in (56) are satisfied for $d=5$, and $b < 13/12$.

For the second term on the right hand side of (53), we follow the same lines than (35)-(47), now taking advantage of the fact that the volume of $\mathcal{D}_{i,k}^{(l)}$ is small. As in (35), for fixed l and k, we have now

$$\mathbb{P}_0\left[J^{(l)} > u; \forall i,\, |\mathcal{D}_{i,k}^{(l)}| \leq 2^{N(1-\gamma_i)}\right] \leq M \exp\left(-\frac{\kappa_d u}{M} 2^{-N\max(b_{i+1}+(1-\gamma_i)2/d)}\right).$$

With the previous choice for the γ_i, this yields

$$\mathbb{P}_0\left[J^{(l)} > u; \forall i,\, |\mathcal{D}_{i,k}^{(l)}| \leq 2^{N(1-\gamma_i)}\right] \leq \exp(-C2^{-N\xi}u) \text{ with } \xi = b(1-\frac{2}{d}-\frac{4}{d^2})+\frac{2}{d}+\delta.$$

Therefore, $\forall \lambda > 0$,

$$\mathbb{P}_0\left[\sum_{k=1}^{2^{l-1}} \bar{J}_k^{(l)} > \frac{2^N y}{N}, \mathcal{A}\right] \leq \exp(-\lambda 2^N y/N)\mathbb{E}_0\left[\exp(\lambda J^{(l)}); \forall i, |\mathcal{D}_{i,k}^{(l)}| \leq 2^{N(1-\gamma_i)}\right]^{2^{l-1}}.$$

Following the same lines than (44)-(47), we end up with

$$\mathbb{P}_0\left[\sum_{k=1}^{2^{l-1}} \bar{J}_k^{(l)} > \frac{2^N y}{N}, \mathcal{A}\right] \leq \exp(-C2^{N(1-\xi)}).$$

Now $1-\xi > 1/2$ if $b < \frac{d(d-4)}{2(d^2-2d-4)}$, and for $d=5$, $1/d < \frac{d(d-4)}{2(d^2-2d-4)}$. ■

4.2 Proof of 2. of Proposition 1.5

Since

$$\begin{aligned}
\mathbb{P}_0\left[\sum_{x:\, l_n(x)\geq\sqrt{n}} l_n^2(x) \geq ny\right] &\leq \mathbb{P}_0\left[\exists x; l_n(x) \geq \sqrt{n}\right] \leq \sum_{x\in]-n;n[^d} \mathbb{P}_0\left[l_n(x) \geq \sqrt{n}\right] \\
&\leq \sum_{x\in]-n;n[^d} \mathbb{P}_0(H_x < \infty)\mathbb{P}_x(l_n(x) \geq \sqrt{n}) \\
&\leq cn^d\mathbb{P}_0(l_n(0) \geq \sqrt{n}) \leq cn^d \exp(-\underline{c}\sqrt{n}),
\end{aligned}$$

it is enough to prove that for $d \geq 5$, for any $y > 0$ and any $\epsilon \in]0, 1/2-1/d[$, $\exists \tilde{c} > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_0\left[\sum_{x:\, n^{1/2-\epsilon}<l_n(x)\leq\sqrt{n}} l_n^2(x) \geq ny\right] \leq -\tilde{c}. \tag{57}$$

We write again $\left\{x : n^{1/2-\epsilon} < l_n(x) \leq \sqrt{n}\right\} \subset \cup_{i=0}^{M-1}\mathcal{D}_i$, $b_0 \leq 1/2-\epsilon$, $b_M = 1/2$, but this time, M will depend on n (actually $M \simeq \log(\log(n))$. Let $(y_i, i = 0\cdots M-1)$ be positive numbers such that $\sum_i y_i \leq 1$. Then,

$$
\begin{aligned}
&\mathbb{P}_0\left[\sum_{x:\, n^{1/2-\epsilon}<l_n(x)\leq\sqrt{n}} l_n^2(x) \geq ny\right] \\
&\leq \sum_{i=0}^{M-1} \mathbb{P}_0\left[\sum_{x\in\mathcal{D}_i} l_n^2(x) \geq ny_iy\right] \leq \sum_{i=0}^{M-1} \mathbb{P}_0\left[|\mathcal{D}_i| \geq n^{1-2b_{i+1}} y_iy\right] \\
&\leq \sum_{i=0}^{M-1} n^{dn^{1-2b_{i+1}}y_iy} \exp\left(-\kappa_d n^{b_i-(1-2/d)(1-2b_{i+1})}(y_iy)^{1-2/d}\right) ,
\end{aligned}
$$

by Lemma 3.1. Therefore, we need to choose $(y_i, b_i, 0 \leq i \leq M-1)$ such that for some $\beta > 0$,

$$
\left\{\begin{array}{l} n^{1-2b_{i+1}} y_i \log(n) \ll n^{b_i-(1-2/d)(1-2b_{i+1})} y_i^{1-2/d} \\ n^{b_i-(1-2/d)(1-2b_{i+1})} y_i^{1-2/d} \geq \beta\sqrt{n} \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} (n^{1-2b_{i+1}} y_i)^{2/d} \log(n) \ll n^{b_i} \\ \beta n^{1/2-b_i} \leq n^{2(1-2/d)(1/2-b_{i+1})} y_i^{1-2/d} \end{array}\right. \tag{58}
$$

For $i = M-1$, the second condition in (58) is $\beta n^{1/2-b_{M-1}} \leq y_{M-1}^{1-2/d}$, so that we have to take $1/2 - b_{M-1} = 1/\log(n)$, and $y_{M-1} = (\beta e)^{\frac{d}{d-2}}$. For this choice of b_{M-1}, y_{M-1}, the first condition in (58) is satisfied.

For the others b_i ($i \leq M-2$), we take $b_{i+1} - 1/2 = a(b_i - 1/2)$, with $\frac{d}{2(d-2)} < a < 1$. Hence for $i \leq M-1$, $\frac{1}{2} - b_i = (\frac{1}{a})^{M-1-i}\frac{1}{\log(n)}$. If we want $b_0 \leq \frac{1}{2} - \epsilon < a$, we have now to take $M - 1 = \lceil\frac{\log(\epsilon\log(n))}{\log(1/a)}\rceil$. With these choices, the second condition in (58) becomes for $i \leq M-2$, $y_i \geq \beta^{\frac{d}{d-2}} \exp\left(-2(1/a)^{M-i-1}(a - \frac{d}{2(d-2)})\right)$, and we take y_i to satisfy the equality. Now, the first condition in (58) is for $i \leq M-2$,

$$
\beta^{\frac{2}{d-2}} \exp\left(\frac{d}{d-2}\left(\frac{1}{a}\right)^{M-i-1}\right) \ll \frac{\sqrt{n}}{\log(n)} \Leftarrow \beta^{\frac{2}{d-2}} \exp\left(\frac{d}{d-2}\left(\frac{1}{a}\right)^{M-1}\right) \ll \frac{\sqrt{n}}{\log(n)} . \tag{59}
$$

Recalling the value of M, this is satisfied as soon as

$$
\frac{\epsilon}{a}\left(\frac{d}{d-2}\right) < \frac{1}{2} . \tag{60}
$$

But for $\epsilon < 1/2 - 1/d$, one can find $a \in]\frac{d}{2(d-2)}, 1[$ such that (60) holds.

It remains now to check that we can take β in order to get $\sum_{i=1}^{M-1} y_i \leq 1$. But,

$$
\begin{aligned}
\sum_{i=1}^{M-1} y_i &= \beta^{\frac{d}{d-2}}\left[e^{\frac{d}{d-2}} + \sum_{i=1}^{M-1} \exp\left(-2\left(a - \frac{d}{2(d-2)}\right)\left(\frac{1}{a}\right)^i\right)\right] \\
&\leq \beta^{\frac{d}{d-2}}\left[e^{\frac{d}{d-2}} + \sum_{i=1}^{\infty} \exp\left(-2\left(a - \frac{d}{2(d-2)}\right)\left(\frac{1}{a}\right)^i\right)\right] .
\end{aligned}
$$

Since the last series is convergent, one can obviously find β such that $\sum_{i=0}^{M-1} y_i \leq 1$. ■

5 Upper bounds for the deviations of the RWRS

The aim of this Section is to prove Proposition 1.1. Let Λ denote the log-Laplace transform of $\eta(0)$:

$$\forall t \in \mathbb{R}\,, \Lambda(t) = \log E_\eta\left[\exp(t\eta(0))\right]\,.$$

Since $\eta(0)$ is centered, there exists a constant C_0 such that for $|t| \leq 1$, $\Lambda(t) \leq C_0 t^2$. By Tauberian Theorem, for $\eta(0)$ having the tail behavior (4), $\Lambda(t)$ is of order t^{α^*} for large t, where α^* is the conjugate exponent of α ($\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1$). Hence, there exists a constant C_∞ such that for $t \geq 1$, $\Lambda(t) \leq C_\infty t^{\alpha^*}$.

Let b be any positive real, and let as usual $\bar{\mathcal{D}}_b = \left\{x \in \mathbb{Z}^d; l_n(x) \geq n^b\right\}$ and $\underline{\mathcal{D}_b} = \left\{x \in \mathbb{Z}^d; l_n(x) \leq n^b\right\}$. Then, for all $y_1, y_2 > 0$, such that $y_1 + y_2 = y$,

$$P\left[\sum_x \eta(x) l_n(x) \geq n^\beta y\right] \leq P\left[\sum_{x\in\bar{\mathcal{D}}_b} \eta(x) l_n(x) \geq n^\beta y_1\right] + P\left[\sum_{x\in\underline{\mathcal{D}_b}} \eta(x) l_n(x) \geq n^\beta y_2\right]. \quad (61)$$

Let A be the event $\left\{\sum_{x\in\bar{\mathcal{D}}_b} l_n^{\alpha^*}(x) \geq n^{\beta-b+\alpha^* b}\frac{y_1}{2C_\infty}\right\}$.

$$\begin{aligned} P\left[\sum_{x\in\bar{\mathcal{D}}_b} \eta(x) l_n(x) \geq n^\beta y_1\right] &\leq \mathbb{P}_0\left[A\right] + P\left[A^c\,;\, \sum_{x\in\bar{\mathcal{D}}_b} \eta(x) l_n(x) \geq n^\beta y_1\right] \\ &\leq \mathbb{P}_0\left[A\right] + \exp(-n^{\beta-b} y_1)\mathbb{E}_0\left[\mathbb{1}_{A^c} \exp\left(\sum_{x\in\bar{\mathcal{D}}_b} \Lambda\left(\frac{l_n(x)}{n^b}\right)\right)\right]. \end{aligned}$$

Now, on $\bar{\mathcal{D}}_b$, $l_n(x) \geq n^b$, so that using the behaviour of Λ near infinity,

$$\begin{aligned} P\left[\sum_{x\in\bar{\mathcal{D}}_b} \eta(x) l_n(x) \geq n^\beta y_1\right] &\leq \mathbb{P}_0\left[A\right] + e^{-n^{\beta-b}y_1}\mathbb{E}_0\left[\mathbb{1}_{A^c} \exp\left(C_\infty \frac{\sum_{x\in\bar{\mathcal{D}}_b} l_n^{\alpha^*}(x)}{n^{\alpha^* b}}\right)\right] \quad (62) \\ &\leq \mathbb{P}_0\left[\sum_{x\in\bar{\mathcal{D}}_b} l_n^{\alpha^*}(x) \geq n^{\beta-b+\alpha^* b}\frac{y_1}{2C_\infty}\right] + \exp(-n^{\beta-b}y_1/2) \quad (63) \end{aligned}$$

Exactly in the same way, but using this time the behiavour of Λ near 0,

$$P\left[\sum_{x\in\underline{\mathcal{D}_b}} \eta(x) l_n(x) \geq n^\beta y_2\right] \leq \mathbb{P}_0\left[\sum_{x\in\underline{\mathcal{D}_b}} l_n^2(x) \geq n^{\beta+b}\frac{y_2}{2C_0}\right] + \exp(-n^{\beta-b}y_2/2)\,. \quad (64)$$

Proposition 1.1 is now a consequence of the following two Lemmas.

Lemma 5.1 *Let $d \geq 3$, $b > 0$, $y > 0$, $\alpha > 1$, and $\beta > 0$ be such that $\beta \leq (1+\alpha)b$, $\beta < (1+\frac{d}{2})b$. Then there exists a constant C such that for large enough n,*

$$\mathbb{P}_0\left[\sum_{x\in\bar{\mathcal{D}}_b} l_n^{\alpha^*}(x) \geq n^{\beta-b+\alpha^* b} y\right] \leq \exp(-Cn^{\beta-b})\,.$$

Lemma 5.2 *Assume $d \geq 5$. Choose positive b, y, β satisfying $\beta + b \geq 1$,*

$$\beta < \min\left((1+\frac{d}{2})b; \frac{b(d^2+d-2)-d}{d-2}\right); \quad \text{and} \quad \begin{cases} y > 0 & \text{if } \beta+b>1; \\ y > 1+2\sum_x G_d^2(x) & \text{if } \beta+b=1\,. \end{cases}$$

Then, there is a constant C so that for large enough n

$$\mathbb{P}_0\left[\sum_{x\in\underline{\mathcal{D}_b}} l_n^2(x) \geq n^{\beta+b}y\right] \leq \exp(-Cn^{\beta-b})$$

Suppose now that Lemma 5.1 and 5.2 hold.

Region I. Choose $b = 1-\beta$, $y_1 > 0$, $y_2 > 2(1+2\sum_x G_d^2(x))C_0$ in (63) and (64). In order to apply Lemmas 5.1 and 5.2, β has to verify

$$\begin{cases} \beta \leq (1+\alpha)(1-\beta) \\ \beta < (1+\frac{d}{2})(1-\beta) \\ \beta(d-2) < (1-\beta)(d^2+d-2)-d \end{cases} \Leftrightarrow \begin{cases} \beta \leq \frac{\alpha+1}{\alpha+2} \\ \beta < \frac{d+2}{d+4} \\ \beta < \frac{d^2-2}{d^2+2d-4} \end{cases}.$$

Note that $\frac{d^2-2}{d^2+2d-4} \leq \frac{d+2}{d+4}$. Therefore, if $\beta \leq \frac{\alpha+1}{\alpha+2}$ and $\beta < \frac{d^2-2}{d^2-2d-4}$, Lemmas 5.1 and 5.2, and equations (61), (63) and (64) lead to, for all $y = y_1 + y_2 > 2MC_\infty$, and large enough n,

$$P\left[\sum_x \eta(x)l_n(x) \geq n^\beta y\right] \leq \exp(-Cn^{\beta-b})\,.$$

This is (6) of Proposition 1.1, since $\beta - b = 2\beta - 1$.

Region II. Choose $b = \frac{\beta}{\alpha+1}$, $y_1 > 0$, $y_2 > 0$ in (63) and (64). In order to apply Lemmas 5.1 and 5.2, β has now to verify

$$\begin{cases} \beta + b > 1 \\ \beta \leq (1+\alpha)b \\ \beta < (1+\frac{d}{2})b \\ \beta(d-2) < b(d^2+d-2)-d \end{cases} \Leftrightarrow \begin{cases} \beta > \frac{\alpha+1}{\alpha+2} \\ \alpha < \frac{d}{2} \\ \beta > \frac{d(\alpha+1)}{d^2-\alpha d+2\alpha} \end{cases}.$$

Under these conditions, Lemmas 5.1 and 5.2, and equations (61), (63) and (64) lead to, for all $y = y_1 + y_2 > 0$, and large enough n,

$$P\left[\sum_x \eta(x)l_n(x) \geq n^\beta y\right] \leq \exp(-Cn^{\beta-b})\,.$$

This is (7) of Proposition 1.1, since $\beta - b = \beta\alpha/(\alpha+1)$. ■

Proof of Lemma 5.1: We apply Lemma 3.2 with

$$p = \alpha^*, \quad \gamma = \beta - b + \alpha^* b, \quad \zeta = \beta - b.$$

Note that the condition $b \geq 2\zeta/d$ is equivalent to $\beta \leq (1+d/2)b$.

Case $\alpha > \frac{d}{2}$: We have $p < (d/2)^*$. The condition $b > \frac{d\zeta - \gamma(d-2)}{d-(d-2)\alpha^*}$ is equivalent to $\beta < (1+d/2)b$. It only remains to invoke Lemma 3.2 (ii).

Case $\alpha \leq \frac{d}{2}$: We have $p \geq (d/2)^*$. The condition $\zeta < \gamma d/(2\alpha^* + d)$ is equivalent to $\beta < (1+d/2)b$. Hence, Lemma 3.2 allows to conclude that for all $y > 0$, for $a < (\gamma(d-2) - d\zeta)/((d-2)\alpha^* - d)$, and n sufficiently large,

$$\mathbb{P}_0\left[\sum_{x: n^b \leq l_n(x) \leq n^a} l_n^{\alpha^*}(x) \geq n^{\beta - b + \alpha^* b} y\right] \leq \exp(-Cn^{\beta - b}).$$

On the other side, for all $y > 0$,

$$\mathbb{P}_0\left[\sum_{x: l_n(x) \geq n^{\beta - b}} l_n^{\alpha^*}(x) \geq n^{\beta - b + \alpha^* b} y\right] \leq \mathbb{P}_0\left[\exists x \in]-n; n[^d; l_n(x) \geq n^{\beta - b}\right] \leq \exp(-Cn^{\beta - b}).$$

Hence, we are left with $\beta - b < (\gamma(d-2) - d\zeta)/((d-2)\alpha^* - d)$, which is equivalent to $\beta < (\alpha+1)b$.

It remains now to treat the case $\beta = (\alpha+1)b$. In that case, $\beta - b = \alpha b$, $\beta - b + \alpha^* b = (\alpha + \alpha^*)b = \alpha\alpha^* b$. Lemma 3.2 allows to conclude that for all $y > 0$, and all $\epsilon > 0$,

$$\mathbb{P}_0\left[\sum_{x: n^b \leq l_n(x) \leq n^{\alpha b - \epsilon}} l_n^{\alpha^*}(x) \geq n^{\alpha\alpha^* b} y\right] \leq \exp(-Cn^{\alpha b}).$$

Hence, it remains to prove that for all $y > 0$, all $\epsilon > 0$, and n sufficiently large,

$$\mathbb{P}_0\left[\sum_{x: n^{\alpha b - \epsilon} \leq l_n(x) \leq n^{\alpha b}} l_n^{\alpha^*}(x) \geq n^{\alpha\alpha^* b} y\right] \leq \exp(-Cn^{\alpha b}).$$

We are in exactly the same situation than in point 2. of Proposition 1.5. The proof is the same, and is left to the reader.

■

Proof of Lemma 5.2. We begin by applying Lemma 3.2 with $p = 2$, $\gamma = \beta + b$, $\zeta = \beta - b$, and $a = b$. The conditions $\zeta < (\gamma d)/(2p + d)$ and $a < (\gamma(d-2) - d\zeta)/((d-2)p - d)$ are both equivalent to $\beta < (1 + \frac{d}{2})b$. Therefore, if $\beta < (1 + \frac{d}{2})b$, Lemma 3.2 yields that $\forall y > 0$, and for n large enough,

$$\mathbb{P}_0\left[\sum_{x: n^{2(\beta - b)/d} \leq l_n(x) \leq n^b} l_n^2(x) \geq n^{\beta + b} y\right] \leq \exp(-Cn^{\beta - b}). \tag{65}$$

We apply now Lemma 3.3 with $\gamma = \beta + b$, $\zeta = \beta - b$, and consider only sites where the local time satisfies $l_n(x) \leq n^{2(\beta - b)/d}$. The second condition of (30) in Lemma 3.3 is equivalent to $\beta(d-2) < b(d^2+d-2)-d$. Thus, for $\beta+b \geq 1$, $\beta(d-2) < b(d^2+d-2)-d$, and all $y>0$ if $\beta+b>1$ or $y>M$ if $\beta+b=1$, we obtain by Lemma 3.3 that for n large enough,

$$\mathbb{P}_0\left[\sum_{x:l_n(x)\leq n^{2(\beta-b)/d}} l_n^2(x) \geq n^{\beta+b}y\right] \leq \exp(-Cn^{\beta-b})\,. \tag{66}$$

Putting together (65) and (66), we conclude Lemma 5.1. ■

6 Lower Bounds for the RWRS.

This Section is devoted to the proof of Proposition 1.3. The symmetry assumption simplifies the proof, thanks to the following Lemma

Lemma 6.1 *(Lemma 2.1 of [2]) When $\{\eta(x), x\in\mathbb{Z}^d\}$ are independent and have bell-shaped densities, then for any Λ finite subset of $\mathbb{Z}^d$, and any $y>0$*

$$P\left(\sum_{x\in\Lambda}\alpha_x\eta(x)>y\right) \leq P\left(\sum_{x\in\Lambda}\beta_x\eta(x)>y\right), \quad \textit{if} \quad 0\leq\alpha_x\leq\beta_x \textit{ for all } x\in\Lambda. \tag{67}$$

Region I. Let us denote by $\mathcal{R}_n$ the range of the random walk

$$\mathcal{R}_n = \{x; l_n(x)\geq 1\}\,.$$

Under the symmetry assumption, $\forall c>0$,

$$P\left[\sum_x \eta(x)l_n(x)\geq n^\beta y\right] \geq P\left[\sum_{x\in\mathcal{R}_n}\eta(x)\geq n^\beta y\right] \geq \mathbb{P}_0(|\mathcal{R}_n|\geq cn)P_\eta\left[\sum_{j=1}^{cn}\eta_j\geq n^\beta y\right]\,.$$

Now, it is well known, that for $d\geq 3$, there is $c>0$ such that $\lim_{n\to\infty}\mathbb{P}_0(|\mathcal{R}_n|\geq cn)=1$. For the other terms, if $1/2<\beta<1$, we are in a regime of moderate deviations for a sum of i.i.d., and there is $C>0$ such that

$$\liminf_{n\to\infty}\frac{1}{n^{2\beta-1}}\log P_\eta\left[\sum_{j=1}^{cn}\eta_j\geq n^\beta y\right]\geq -C.$$

This gives the result for region I.

Region II. Under the symmetry assumption, $\forall c>0$

$$P\left[\sum_x\eta(x)l_n(x)\geq n^\beta y\right]\geq P\left[\eta(0)l_n(0)\geq n^\beta y\right]\geq P_\eta\left[\eta(0)\geq n^{\frac{\beta}{\alpha+1}}y/c\right]\mathbb{P}_0\left[l_n(0)\geq cn^{\frac{\beta\alpha}{\alpha+1}}\right]\,.$$

Now, for $\frac{\beta\alpha}{\alpha+1}\leq 1$, the second probability is of order $\exp(-Cn^{\frac{\beta\alpha}{\alpha+1}})$, which is also the order of the first one. This leads to the lower bound in region II.

■

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